

Quantum phenomena in terms of energy-momentum transfer

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1999 J. Phys. A: Math. Gen. 32 3069

(<http://iopscience.iop.org/0305-4470/32/16/014>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.105

The article was downloaded on 02/06/2010 at 07:29

Please note that [terms and conditions apply](#).

Quantum phenomena in terms of energy–momentum transfer

Peter J Riggs

Department of Physics, University of Queensland, Brisbane, Queensland, Australia

Received 22 May 1998

Abstract. In the deBroglie–Bohm causal interpretation of quantum mechanics, the motion of a quantum particle is governed by its wave field which is taken to be a physically real field. The surrounding environment modifies the form of the wave field which, in turn, alters the motion of the particle. It is found that the dynamic role of the wave field is to influence the trajectory of a quantum particle by means of transferring energy–momentum.

1. Introduction

In the deBroglie–Bohm causal interpretation of quantum mechanics, a quantum particle has a well-defined position at all times. This does not conflict with the Heisenberg uncertainty relations for these refer to the statistical scatter obtained for the *measured values* of complementary variables in an ensemble of similarly prepared systems. A quantum particle's motion is governed by its accompanying wave field which is interpreted as an objectively existing field that propagates according to the Schrödinger equation. The single-particle wavefunction expressed in polar form is: $\Psi(x, t) = R e^{iS/\hbar}$, where R and S are real functions of the space–time coordinates and $R \geq 0$. If this expression for Ψ is substituted into the time-dependent Schrödinger equation for a (spinless) particle of mass m , the result is the following two real differential equations:

$$-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V + Q \quad (1)$$

and

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{\nabla S}{m} \right) = 0 \quad (2)$$

where $(\nabla S)^2/2m$ is the particle's kinetic energy (T), V is a classical potential, $\rho = R^2$ and $Q = (-\hbar^2/2m)(\nabla^2 R/R)$ is called the quantum potential. Q is a physical potential which accounts for many of the differences between classical and quantum physics. Equation (1) is known as the quantum Hamilton–Jacobi equation whilst equation (2) is a continuity equation for the quantity ρ which is interpreted as a probability density. The total energy of a quantum particle (E) is given by $(-\partial S/\partial t)$ with momentum (∇S) . Equation (1) describes an ensemble of particles, the trajectories of which are all normal to surfaces of constant S [1]. The trajectory of an individual quantum particle can then be found if its initial position is specified. Such detail is specifically excluded in the Copenhagen interpretation of quantum mechanics.

2. The wave field and the quantum potential

The wave field and its particle are physically inseparable aspects of a single quantum entity (i.e. a one-particle system). This, however, does not prevent an *in principle* analysis of the causal role exhibited by the wave field. One surprising property is that the wave field's effects do not depend on its intensity, as can be seen by noting that the magnitude of the quantum potential is unaffected by multiplication of the wave field's amplitude R by a constant. This independence from intensity suggests that the field possesses no *intrinsic* energy–momentum and led Bohm and Hiley to the conclusion that the wave field does *not* transfer energy–momentum [2]. If the wave field did indeed possess energy in its own right then we ought to expect another term in the Hamiltonian operator for a single-particle quantum system, i.e.

$$H = \nabla^2/2m + V + F$$

where $F = F(|\Psi|)$ is a function of the absolute value of the wavefunction [3]. The above Hamiltonian would produce a *nonlinear* Schrödinger-type equation. On this basis, it is reasonable to conclude that the quantum particle (and not the wave field) is the origin of energy–momentum in a single-particle state.

Quantities such as V and Q , whether used in classical or quantum physics, are conventionally defined as potential energies *of the particle*. Under this definition, the total 'particle energy' is just the sum of the kinetic and potential energy terms. However, V and Q are potential energy functions that really represent an energy contained *in a field* [4]. This being the case, the quantum potential would give the potential energy available to the quantum particle at its specific position in the wave field, but need not coincide with the total field energy. Yet, how can this be if the wave field possesses no intrinsic energy–momentum? The answer is that although the wave field is not itself an energy source, it *stores* energy gained from the particle (and also from external interactions when the system is not isolated).

In order to see that this is so, consider an isolated, self-contained, classically free, single-particle quantum system. This would be the case if the quantum entity exists in a region where all other fields are zero and where the particle is not subject to any collisions. The rate of change of the 'particle energy' E with respect to time is found from equation (1):

$$\frac{dE}{dt} = \frac{1}{2m} \frac{d}{dt} (\nabla S)^2 + \frac{dQ}{dt} = \left(\frac{\nabla S}{m} \right) \cdot (-\nabla Q) + \frac{dQ}{dt}. \quad (3)$$

The quantum potential is generally an explicit function of both space and time coordinates, so its total rate of change with respect to time is given by:

$$\frac{dQ}{dt} = \sum_{i=1}^3 \frac{\partial Q}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial Q}{\partial t} = (\nabla Q) \cdot (\nabla S/m) + \frac{\partial Q}{\partial t}.$$

The term $[(\nabla Q) \cdot (\nabla S/m)]$ is equal to minus the rate of change of the particle's kinetic energy with respect to time, i.e. $(-dT/dt)$, as can be seen with reference to equation (3). Substitution for (dQ/dt) yields:

$$\frac{dE}{dt} = (\nabla S) \cdot (-\nabla Q/m) + (\nabla Q) \cdot (\nabla S/m) + \frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial t}. \quad (4)$$

When $(\partial Q/\partial t) = 0$, the 'particle energy' $E = \text{constant}$ because changes in kinetic energy are exactly balanced by changes in the quantum potential. What if $(\partial Q/\partial t) \neq 0$? What does this term represent? Recall that the quantum potential Q gives the potential energy available to the particle at its specific position in the field. In a self-contained single-particle state, the wave field is the only possible *repository* of energy other than the particle itself. This being so, $(\partial Q/\partial t)$ gives the rate of change of the quantum potential due to changes in the amount

of energy stored in the wave field other than at the particle's location. Therefore, the particle's energy will increase (decrease) with decreases (increases) in the amount of energy stored in the wave field. Therefore, the total energy of an isolated, classically-free, single-particle quantum system can be accounted for.

The above explanation could be criticized by claiming that energy is not conserved for the quantum state *as a whole* (i.e. particle and field together). The justification for the non-conservation of energy being that although the wave acts on the particle, the particle does not appear to react back on the wave [5]. This is correct in so far as that the quantum particle does not affect the shape or size of its wave field. However, as emphasized by Cushing [6], the quantum realm need not be bound by classical action–reaction. This being the case, energy could be passed from particle to field under particular circumstances. In a self-contained, classically-free, single-particle quantum system the total energy is conserved because the field energy originates solely from the particle. This is shown as follows. Let $H = \int_{-\infty}^{\infty} \mathcal{H} d^3x$ where \mathcal{H} is the Hamiltonian density of the wave field. In the case of a classically-free system, $\mathcal{H} = R^2(\nabla S)^2/2m + (\hbar^2/2m)(\nabla R)^2$ and for this Hamiltonian density, an integration shows that the value of H is constant [7]. Therefore H cannot be the field energy alone for there are examples of isolated, classically-free systems where the field energy decreases (such as a Gaussian wavepacket described below). However, if the system is isolated, H must be the *total* energy since only total energy is conserved. Let the energy content of the wave field (other than that given by the quantum potential) in a non-stationary state be U , then $U = H - (T + Q)$. Consequently,

$$\frac{dU}{dt} = \frac{dH}{dt} - \left(\frac{dT}{dt} + \frac{dQ}{dt} \right) = -\frac{dE}{dt} = -\frac{\partial Q}{\partial t} \quad (5)$$

by equation (4). Therefore, as concluded above, in the isolated, classically-free case, changes in the energy content of the wave field appear as changes in the quantum potential.

3. Energy exchanges in individual quantum processes

There is a fundamental difference between the wave field and classical fields for the capacity of the wave field to store energy depends, not on its intensity, but on the shape of the field. Consider a free (spinless) particle of mass m that is initially not subject to any force fields or barriers. The particle moves with a constant velocity and its corresponding wave field is represented by a plane wave. Since $\nabla^2 R = 0$ for a plane wave, the value of its quantum potential is zero. An application of Gauss' law shows that $H = \langle E \rangle$, the energy expectation value [8]. This is just equal to the 'particle energy' E , as the system is in an energy eigenstate. Since both V and Q are zero, $E = T$, the particle's kinetic energy. Since H is the total energy in an isolated, classically-free system, the field energy U must be zero. If the shape is then altered the wave field will, in general, store some energy. The shape is determined, in large part, on whether the wave field has encountered any obstructions which have distorted it. Depending on the prevailing circumstances, some (or all) of a particle's energy–momentum can be transferred and temporarily stored in its wave field. Once stored in the field, energy–momentum can be returned to the particle if circumstances change. The particle's trajectory, therefore, need not be in a straight line even if there is no external field present. Since the particle is inseparable from its 'guiding' wave field, exchanges of energy–momentum occur between wave field and particle as they move along together.

We need look no further for an example which shows that the wave field is a repository of potential energy gained from its quantum particle than the simple case of an infinite well with a zero classical potential inside. If a free (spinless) particle were suddenly to become confined

in a cubical well of side length L , then its wavefunction would be altered from that given by a plane wave to the stationary form:

$$\Psi = (2/L)^{3/2} |\sin(n_1\pi x/L) \sin(n_2\pi y/L) \sin(n_3\pi z/L)| e^{-iE_n t/\hbar} = \text{Re}^{iS/\hbar}$$

with total energy $E = (n_1^2 + n_2^2 + n_3^2)(\pi^2\hbar^2/2mL^2)$, where the n are positive integers. In the Copenhagen interpretation, if the existence of the particle is acknowledged at all, then it is taken to have kinetic energy only (as $V = 0$ inside the well) and to be bouncing back and forth between (perfectly reflecting) impenetrable walls. In the causal interpretation, the situation is seen to be very different. The quantum potential is now:

$$Q = -(\hbar^2/2m)(\nabla^2 R)/R = (n_1^2 + n_2^2 + n_3^2)(\pi^2\hbar^2/2mL^2) \quad (6)$$

which is the *same* magnitude as the particle's kinetic energy in the Copenhagen interpretation. Since $S = -Et$, $\nabla S = 0$, i.e., the particle's momentum is zero! The particle is at rest because all its energy has been stored in the wave field. This explanation was originally suggested by Bohm [9]. One objection denies this account on the grounds that the field amplitude varies from a maximum at its anti-nodes to zero at its nodes, whereas Q in equation (6) is independent of position. This objection is ill-founded for, in a stationary state, the quantum potential does not represent the potential energy at a specific location but gives the value of the total field energy.

A possible experimental test of whether a trapped, massive, spinless particle (or an atom) is at rest or not might be conducted using the new techniques of atom optics. A measurement that would not disturb the wave field is necessary as such a disturbance would cause the particle (atom) to accelerate. The particle (atom) could be 'laser cooled' and placed in a containment vessel with evanescent-light wave reflectors at each end. It should be possible to determine if the particle (atom) is in motion by observing if there is a phase shift in the evanescent light (see [10] for further details).

In many real situations such as diffraction, one would expect the wave field to be initially of a small width and localized about the particle. This is described mathematically by a wavepacket with the quantum particle located somewhere within the packet. A Gaussian wavepacket can be used to model phenomena such as diffraction by a slit with imperfect edges [11]. At a large distance from the slit, an incoming quantum particle has a wavefunction represented by a plane wave which has zero field energy. Upon passing through the slit, the shape of the wave field is altered to that represented by a Gaussian wavepacket and in the process, energy-momentum is transferred from the particle to the wave field. The form of a Gaussian wavepacket is given by:

$$\Psi(\mathbf{x}, t) = (2\pi s_t^2)^{-3/4} \exp\{i\mathbf{k} \cdot (\mathbf{x} - \frac{1}{2}\mathbf{u}t) - (\mathbf{x} - \mathbf{u}t)^2/4\sigma_0 s_t\} \quad (7)$$

where σ_0 is the initial root-mean-square (RMS) width of the packet in each coordinate direction, where $\sigma_0^2 = \langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle$, $s_t = \sigma_0(1 + i\hbar t/2m\sigma_0^2)$ and \mathbf{u} is the initial (group) velocity. In the diffraction case, the wavepacket will expand and change its shape as the particle moves away from the slit. The particle then will act as a 'sink' for energy stored in the field, but the total energy of the system (particle and field) will remain constant. Note that both energy and momentum pass to the quantum particle from the wave field. The particle will thereby accelerate until such time as the value of the quantum potential effectively drops to zero.

This can be readily demonstrated with reference to the Gaussian wavefunction (equation (7)). The corresponding quantum potential is:

$$Q = (\hbar^2/4m\sigma^2)\{3 - (\mathbf{x} - \mathbf{u}t)^2/2\sigma^2\}$$

where $\sigma = |s_t| = \sigma_0[1 + (\hbar t/2m\sigma_0^2)^2]^{1/2}$ is the RMS width of the packet at time t [12]. The partial time rate of change of the quantum potential and the rate of change of the particle's

momentum with respect to time are respectively:

$$\frac{\partial Q}{\partial t} = \frac{\hbar^4 t}{8m^3 \sigma_0^2 \sigma^6} (x - ut)^2 + \frac{\hbar^2}{4m\sigma^4} [\mathbf{u} \cdot (x - ut)] - \frac{3\hbar^4 t}{8m^3 \sigma_0^2 \sigma^4}$$

and

$$\frac{d\mathbf{p}}{dt} = -(\nabla Q) = \frac{\hbar^2}{4m\sigma^4} (x - ut).$$

Consider a quantum particle positioned in front of its wavepacket, so that $(x - ut) > 0$, then $(d\mathbf{p}/dt)$ and $(\partial Q/\partial t)$ will be positive. $(\partial Q/\partial t)$ is a measure of the rate at which the value of Q is ‘topped-up’ from the rest of the wave field. The positive value of $(d\mathbf{p}/dt)$ shows that the particle’s momentum is increasing, as will its kinetic energy. This is confirmed by the rate of change of the particle’s kinetic energy with respect to time:

$$(dT/dt) = [-(\nabla Q) \cdot (\nabla S/m)] = (\hbar^2/4m\sigma^4)[\mathbf{u} \cdot (x - ut)] + (\hbar^4 t/16m^3 \sigma_0^2 \sigma^6)(x - ut)^2 > 0$$

where $\nabla S = m\mathbf{u} + (\hbar^2 t/4m\sigma_0^2 \sigma^2)(x - ut)$. The dominant terms for large t will be those containing powers of σ . Since the σ are denominator terms, Q , $(\partial Q/\partial t)$, $(d\mathbf{p}/dt)$ and (dT/dt) all will tend to zero as $t \rightarrow \infty$. Provided no further obstacles, disturbances or fields are encountered, it is now clear that the wave field will expand extensively with time. Correspondingly, the quantum potential and the energy contained in the wave field as a whole will drop rapidly, resulting in the energy of the quantum system becoming overwhelmingly kinetic.

An expression for the time taken for energy to be transferred to the quantum particle can be evaluated in the case of a classically-free, expanding Gaussian wavepacket. The particle’s kinetic energy at time t is:

$$T = (\nabla S)^2/2m = \frac{1}{2}m|\mathbf{u}|^2 + (\hbar^2 t/4m\sigma_0^2 \sigma^2)[\mathbf{u} \cdot (x - ut)] + (\hbar^4 t^2/32m^3 \sigma_0^4 \sigma^4)(x - ut)^2.$$

Thus

$$\begin{aligned} (\sigma_0/\sigma)^2(T - T_i) &= (\hbar^2 t/4m\sigma^4)[\mathbf{u} \cdot (x - ut)] + (\hbar^4 t^2/32m^3 \sigma_0^2 \sigma^6)(x - ut)^2 \\ &= t(dT/dt) - (\hbar^4 t^2/32m^3 \sigma_0^2 \sigma^6)(x - ut)^2 \end{aligned}$$

where $T_i = \frac{1}{2}m|\mathbf{u}|^2$ is the initial kinetic energy of the particle.

Now

$$\begin{aligned} (\hbar^4 t^2/32m^3 \sigma_0^2 \sigma^6)(x - ut)^2 &= (\sigma_0/\sigma)^2(T - T_i) - (\hbar^2 t/4m\sigma^4)[\mathbf{u} \cdot (x - ut)] \\ \Rightarrow (\sigma_0/\sigma)^2(T - T_i) &= t(dT/dt) - \{(\sigma_0/\sigma)^2(T - T_i) - (\hbar^2 t/4m\sigma^4)[\mathbf{u} \cdot (x - ut)]\}. \end{aligned}$$

Or

$$2(\sigma_0/\sigma)^2(T - T_i) = t(dT/dt) + t\mathbf{u} \cdot (d\mathbf{p}/dt).$$

Rearranging and integrating gives:

$$2\sigma_0^2 \int \frac{dt}{\sigma^2 t} = \int \frac{dT}{(T - T_i)} + \int \frac{\mathbf{u} \cdot d\mathbf{p}}{(T - T_i)} = \int \frac{dT}{(T - T_i)} + (\sqrt{2m}|\mathbf{u}|\varepsilon) \int \frac{d\sqrt{T}}{(T - T_i)}$$

where $\varepsilon = (\cos \theta / \cos \phi)$, with $\cos \theta = (\mathbf{u} \cdot d\mathbf{p})/(|\mathbf{u}||d\mathbf{p}|)$, $\cos \phi = (\mathbf{p} \cdot d\mathbf{p})/(|\mathbf{p}||d\mathbf{p}|)$, $|d\mathbf{p}| = d|\mathbf{p}|/\cos \phi$ and $|\mathbf{p}| = \sqrt{2mT}$. This results in the following expression:

$$\log \left| \frac{t^2}{(\hbar^2 t^2/4m^2 \sigma_0^4) + 1} \right| + A = \log |T - T_i| + \varepsilon \log \left| \frac{\sqrt{T} - \sqrt{T_i}}{\sqrt{T} + \sqrt{T_i}} \right|$$

where A is a constant of integration. In order to ensure consistency of left- and right-hand sides of the above equation, the value of A is set to zero. Taking exponentials gives:

$$\frac{Ct^2}{(\hbar^2 t^2 / 4m^2 \sigma_0^4) + 1} = (T - T_i) \left(\frac{\sqrt{T} - \sqrt{T_i}}{\sqrt{T} + \sqrt{T_i}} \right)^\varepsilon$$

where $C = e^A = 1$. The effect of the ε term is to vary the time taken for the transfer of energy from field to particle, depending on the particle's position in the wavepacket. The time taken for a complete transfer of energy will be when T equals the final kinetic energy of the particle (T_f). If the particle is in a forward and central region of the packet so that $\varepsilon \approx 1$, then for $T = T_f$, the time for transfer is:

$$t = \frac{2m\sigma_0^2(\sqrt{T_f} - \sqrt{T_i})}{\sqrt{4m^2\sigma_0^4 - \hbar^2(T_f + T_i - 2\sqrt{T_f T_i})}}.$$

4. Conclusions

The motion of a quantum particle in the deBroglie–Bohm causal interpretation of quantum mechanics can be explained consistently by a process of energy–momentum transfer between particle and field. This treatment brings out the causal role of the wave field and the physical nature of the quantum potential.

Acknowledgments

The author wishes to thank Peter Szekeres, Gerard Milburn and the anonymous referees for helpful suggestions.

References

- [1] Bohm D 1952 *Phys. Rev.* **85** 170
- [2] Bohm D and Hiley B J 1987 *Phys. Rep.* **144** 323–48
Bohm D and Hiley B J 1993 *The Undivided Universe: An Ontological Interpretation of Quantum Theory* (London/New York: Routledge) pp 31–2
- [3] Selleri F 1990 *Quantum Paradoxes and Physical Reality* (Dordrecht: Kluwer) p 136
- [4] Freistadt H 1957 *Suppl. Nuovo Cimento* **V** 17
Rindler W 1977 *Essential Relativity: Special, General, and Cosmological* (Berlin: Springer) p 83
- [5] Holland P R 1993 *The Quantum Theory of Motion: An Account of the deBroglie–Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press) p 120
- [6] Cushing J T 1994 *Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony* (Chicago: Chicago University Press) p 46
- [7] Holland P R 1993 *The Quantum Theory of Motion: An Account of the deBroglie–Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press) p 116
- [8] Holland P R 1993 *The Quantum Theory of Motion: An Account of the deBroglie–Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press) p 117
- [9] Bohm D 1952 *Phys. Rev.* **85** 184
Bohm D 1953 *Scientific Papers Presented to Max Born* (Edinburgh/London: Oliver and Boyd) p 14
- [10] Milburn G J 1996 *Quantum Technology* (St Leonards, NSW: Allen and Unwin) pp 70–1
- [11] Holland P R 1993 *The Quantum Theory of Motion: An Account of the deBroglie–Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press) p 163
- [12] Holland P R 1993 *The Quantum Theory of Motion: An Account of the deBroglie–Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press) pp 159–61